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Laurence Halpern, Jérémie Szeftel

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Abstract

We introduce a non-overlapping variant of the Schwarz waveform relaxation algorithm for semilinear wave propagation in one dimension. Using the theory of absorbing boundary conditions, we derive a new nonlinear algorithm. We show that the algorithm is well-posed and we prove its convergence by energy estimates and a Galerkin method. We then introduce an explicit scheme. We prove the convergence of the discrete algorithm with suitable assumptions on the nonlinearity. We finally illustrate our analysis with numerical experiments.

1 Introduction

Schwarz waveform relaxation is a new class of algorithms for domain decomposition in the frame of time dependant partial differential equations. They are well-adapted to evolution problems, designed to solve the equations separately on each spatial subdomain on the whole time interval , exchanging informations on the space-time boundary of the subdomains, overlapping or not [6]. In particular for wave equations, it is of great importance, due to numerical dispersion, to be able to handle local time and space meshes, and this is allowed by the present method. We presented the method for the linear wave equation in [7] and [5] . When using overlapping subdomains and "classical" Schwarz waveform relaxation -by a Dirichlet exchange of informations on the boundary- the so defined algorithm converges in a finite number of iterations, inversely proportional to the size of the overlap, which can be penalizing. We introduced optimized transmission conditions, relying on the theory of absorbing boundary conditions, which improve drastically the convergence of the algorithm.

Very little has been done so far about nonlinear Schwarz algorithms. An analysis of the classical Schwarz waveform relaxation algorithm was performed in [4] for a conservation law. The goal of this paper is to define new Schwarz waveform relaxation algorithms for the semilinear wave equation. We introduce two nonoverlapping algorithms. The first one referred to as linear uses the absorbing boundary condition of the linear problem whereas the second one referred to as nonlinear uses the nonlinear absorbing boundary conditions designed by J. Szeftel in [11].

In Section 2, we introduce the definitions of the algorithms.

In Section 3 , we prove the algorithms to be well-posed. For the precise analysis, we use a fixed point algorithm with regularity estimates on a linear problem.

In Section 4, we prove the convergence of the algorithms. The proof is an extension of a clever trick in [8], already used for linear algorithms, either hyperbolic or parabolic (see [7]). However the nonlinearity requires a very fine analysis.

In Section 5, we design discrete Schwarz waveform relaxation algorithms. In each subdomain, the interior scheme is the usual leapfrog scheme for the linear part, with a downwinding in time for the nonlinear part. The exchange of informations on the boundary is naturally taken into account by a finite volume strategy. In Section 6 we study the convergence of the algorithms, by discrete energy estimates.

*LAGA, Institut Galilée, Université Paris XIII, 93430 Villetaneuse, France. halpern@math.univ-paris13.fr

‡Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton NJ 08544-1000, USA and C.N.R.S., Mathématiques Appliquées de Bordeaux, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence cedex France. jszeftel@math.princeton.edu. The second author is partially supported by NSF Grant DMS-0504720

As it is always the case for nonlinear problems, the well-posedness and convergence results hold only locally in time. Therefore numerical experiments are very important to bypass the limitations of the theory. We present the results in Section 7, showing in particular that our nonlinear algorithm gives optimal results within a large class of algorithms.
Remark Due to the complexity of the mathematical theory, we restrain ourselves to the one dimensional case. The multidimensional study contains additional difficulties due to the geometry and should be the heart of a forthcoming paper.

2 Problem Description

We consider the second order semilinear wave equation in one dimension,

$$(\partial_t^2 - \partial_x^2)\mathcal{U} = f(\mathcal{U}, \partial_t \mathcal{U}, \partial_x \mathcal{U}) \tag{1}$$

on the domain $\mathbb{R} \times (0, T)$ with initial conditions $\mathcal{U}(\cdot, 0) = p$, $\partial_t \mathcal{U}(\cdot, 0) = q$.

2.1 Absorbing Boundary Conditions for the Semilinear Wave Equation

The question of absorbing boundary conditions arises when one wants to make computations on an unbounded domain: a bounded computational domain is introduced, on the boundary of which boundary conditions must be prescribed. These boundary conditions must be absorbing to the waves leaving the domain. A whole strategy has been designed by Engquist and Majda for linear problems with variable coefficients, using pseudo-differential operators [3]. Recently it has been extended to nonlinear operators by J. Szeftel, in particular for the semilinear wave equation [11], using the paradifferential calculus of [1] and [9]. We introduce a family of operators

$$\mathcal{B}^{\pm}(g^{\pm})u = \partial_t u \pm \partial_x u + g^{\pm}(u), \tag{2}$$

for C^∞ functions $g^\pm$ such that $g^\pm(0) = 0$. The linear absorbing boundary operators are given by $g^\pm = 0$. In the case where $f(u, u_t, u_x) = f_1(u) + f_2(u)u_t + f_3(u)u_x$ with f_j in $C^\infty(\mathbb{R})$, $1 \leq j \leq 3$, and $f_1(0) = 0$, the following nonlinear boundary operators are given in [11]:

$$g^+(u) := -\frac{1}{2}\int_0^u (f_2 - f_3)(\xi)d\xi, \qquad g^-(u) := -\frac{1}{2}\int_0^u (f_2 + f_3)(\xi)d\xi. \tag{3}$$

We replace the problem on the domain $\mathbb{R}$ by a boundary value problem in $\Omega_0 = (a, b)$:

$$\begin{gathered}(\partial_t^2 - \partial_x^2)\bar{u} = f(\bar{u}, \partial_t \bar{u}, \partial_x \bar{u}) \text{ in } \Omega_0 \times (0, T),\\ \mathcal{B}^-(g^-)\bar{u}(a, \cdot) = 0, \quad \mathcal{B}^+(g^+)\bar{u}(b, \cdot) = 0,\end{gathered} \tag{4}$$

with initial values p and q. Such boundary conditions give well-posed initial boundary value problems, and are absorbing provided the intial data be compactly supported in Ω_0, see [11]. Following the strategy in [7], we use such absorbing operators for domain decomposition.

2.2 A General Non-Overlapping Schwarz Waveform Relaxation Algorithm

We decompose the domain (a, b) into I non overlapping subdomains $\Omega_i = (a_i, a_{i+1})$, $a_j < a_i$ for $j < i$ and $a_1 = a$, $a_{I+1} = b$, and we introduce a general non overlapping Schwarz waveform relaxation algorithm. An initial guess $\{h_i^{\pm,0}\}_{1 \leq i \leq I+1}$ is given. For $k \geq 1$, one step of the algorithm is

$$\left[\begin{array}{l}(\partial_t^2 - \partial_x^2)u_i^k = f(u_i^k, \partial_t u_i^k, \partial_x u_i^k) \text{ in } \Omega_i \times (0, T),\\ u_i^k(\cdot, 0) = p, \quad \partial_t u_i^k(\cdot, 0) = q \text{ in } \Omega_i,\\ \mathcal{B}^-(g^-)u_i^k(a_i, \cdot) = h_i^{-,k-1}, \quad \mathcal{B}^+(g^+)u_i^k(a_{i+1}, \cdot) = h_i^{+,k-1} \text{ in } (0, T),\end{array}\right. \tag{5}$$

$$h_i^{-,k} = \mathcal{B}^-(g^-)u_{i-1}^k(a_i, \cdot), \quad h_i^{+,k} = \mathcal{B}^+(g^+)u_{i+1}^k(a_{i+1}, \cdot) \text{ in } (0, T),$$

where $\mathcal{B}^\pm$ are given in (2). For ease of notations, we defined here $h_1^{\pm,k} = 0$ and $h_{I+1}^{\pm,k} = 0$, so that the index i in (5) ranges from $i = 1, 2, \ldots, I$. In the sequel, we call linear transmission condition the choice $g^\pm = 0$ and nonlinear transmission condition the choice (3). For the classical linear homogeneous wave equation, it has been proved in [7] that the algorithm converges optimally if T is small enough (which means in two iterations, independently of the number of subdomains), and the transmission operators $\mathcal{B}^\pm$ are given by $\mathcal{B}^\pm = \partial_t \pm \partial_x$. This behavior is due to the finite speed of propagation, together with the fact that these operators are the exact Dirichlet Neumann operators in this case. In the nonlinear case, the propagation still takes place with the finite speed, but we can use only approximate Dirichlet Neumann operators. Therefore the classical Schwarz algorithm with overlap is still convergent, and for our nonoverlapping nonlinear algorithms, we will use energy estimates.

3 Well-posedness For The Subproblems

The study of the nonlinear problem relies on an iterative linear scheme. Therefore a first step for the definition of the algorithm is the study of the nonhomogeneous initial boundary value problem for a general domain $\Omega = (a_-, a_+)$,

$$\begin{gathered}(\partial_t^2 - \partial_x^2)u = f(u, \partial_t u, \partial_x u) \text{ in } \Omega \times (0,T),\\ \mathcal{B}^-(g^-)u(a_-,\cdot) = h^-, \quad \mathcal{B}^+(g^+)u(a_+,\cdot) = h^+,\end{gathered} \tag{6}$$

with initial values p and q. We will use for $j \le 2$ the spaces

$$V_j(\Omega,T) = \{u \in L^\infty(0,T;L^2(\Omega)), \partial^\alpha u \in L^\infty(0,T;L^2(\Omega)), |\alpha| \le j\}. \tag{7}$$

In formula (7), α is a 2-index in $\mathbb{N}^2$, the first coordinate in α stands for the time, and the second one stands for the space, so for instance $\partial^\alpha u = \partial_{tx} u$ for $\alpha = (1,1)$. $V_j(\Omega,T)$ is equipped with the norm $\|u\|_{V_j(\Omega,T)} = \max_{|\alpha|\le j} \|\partial^\alpha u\|_{L^\infty(0,T;L^2(\Omega))}$.

Theorem 3.1 *Let p in $H^2(\Omega)$ and q in $H^1(\Omega)$. There exists a time T^* such that for any $T \le T^*$, for $h^\pm$ in $H^1(0,T)$ with the compatibility conditions*

$$h^\pm(0) = q(a_\pm) \pm p'(a_\pm) + g^\pm(p(a_\pm)), \tag{8}$$

(6) has a unique solution u in $V_2(\Omega,T)$, with $\partial_t u(a_\pm,\cdot)$ and $\partial_x u(a_\pm,\cdot)$ in $H^1(0,T)$. Furthermore there exists a positive real number C^* such that

$$\|u\|^2_{V_2(\Omega,T)} + \sum_\pm \sum_{|\alpha|=1} \|\partial^\alpha u(a_\pm,\cdot)\|^2_{H^1(0,T)} \le C^*(\|p\|^2_{H^2(\Omega)} + \|q\|^2_{H^1(\Omega)} + \sum_\pm \|h^\pm\|^2_{H^1(0,T)}), \tag{9}$$

where T^* and C^* depend on the data $p, q, f, g^\pm, h^\pm$.

This result has been first proved in [11] with homogeneous boundary conditions (*i.e.* $h^\pm = 0$). The additional difficulty comes from the boundary conditions, and we give here the main steps of the proof. It relies on the construction of a sequence of linear problems of the form:

$$\begin{gathered}\partial_t^2 \tilde u - \partial_x^2 \tilde u + \tilde u = F \text{ in } \Omega \times (0,T),\\ (\partial_t \tilde u - \partial_x \tilde u)(a_-,\cdot) = H^-, \quad (\partial_t \tilde u + \partial_x \tilde u)(a_+,\cdot) = H^+.\end{gathered} \tag{10}$$

Proposition 3.2 *Let p in $H^2(\Omega)$ and q in $H^1(\Omega)$. For any positive time T, let F in $H^1((0,T)\times\Omega)$, and $H^\pm$ in $H^1(0,T)$ with the compatibility conditions*

$$H^\pm(0) = q(a_\pm) \pm p'(a_\pm). \tag{11}$$

Then, (10) with initial data p and q has a unique solution $\tilde u$ in $V_2(\Omega,T)$, with $\partial_t \tilde u(a_\pm,\cdot)$ and $\partial_x \tilde u(a_\pm,\cdot)$ in $H^1(0,T)$. Moreover we have the following bounds on the solution

$$\begin{aligned}&\|\tilde u\|^2_{V_2(\Omega,T)} + \sum_\pm \big(\|\partial_t \tilde u(a_\pm,\cdot)\|^2_{H^1(0,T)} + \|\partial_x \tilde u(a_\pm,\cdot)\|^2_{H^1(0,T)}\big)\\ &\qquad \le C_1 e^T \big(\|F\|^2_{H^1(0,T;L^2(\Omega))} + \|F(\cdot,0)\|^2_{L^2(\Omega)} + \sum_\pm \|H^\pm\|^2_{H^1(0,T)} + \|p\|^2_{H^2(\Omega)} + \|q\|^2_{H^1(\Omega)}\big),\end{aligned} \tag{12}$$

where C_1 is a universal constant.

Proof We start with the *a priori* estimates. We multiply (10) by $\partial_t \tilde{u}$ and integrate by parts in Ω:

$$\frac{1}{2}\frac{d}{dt}\big(\|\partial_t \tilde{u}(\cdot,t)\|^2_{L^2(\Omega)} + \|\tilde{u}(\cdot,t)\|^2_{H^1(\Omega)}\big) + (\partial_t \tilde{u}(a_-,t))^2 + (\partial_t \tilde{u}(a_+,t))^2$$
$$= (F(\cdot,t), \partial_t \tilde{u}(\cdot,t))_{L^2(\Omega)} + H^-(t)\partial_t \tilde{u}(a_-,t) + H^+(t)\partial_t \tilde{u}(a_+,t).$$

Using the Cauchy-Schwarz inequality on the right hand side, together with the inequality $\alpha\beta \leq \frac{1}{2}\,\alpha^2 + \frac{1}{2}\,\beta^2$ for all $\alpha, \beta \in \mathbb{R}$, and finally integrating in time, we obtain

$$\|\partial_t \tilde{u}(\cdot,t)\|^2_{L^2(\Omega)} + \|\tilde{u}(\cdot,t)\|^2_{H^1(\Omega)} + \int_0^t [(\partial_t \tilde{u}(a_-,s))^2 + (\partial_t \tilde{u}(a_+,s))^2]\, ds$$
$$\leq \int_0^t [\|\partial_t \tilde{u}(\cdot,s)\|^2_{L^2(\Omega)}\, ds + \int_0^t \|F(\cdot,s)\|^2_{L^2(\Omega)}\, ds$$
$$+ \|q\|^2_{L^2(\Omega)} + \|p\|^2_{H^1(\Omega)} + \int_0^t [(H^-(s))^2 + (H^+(s))^2]\, ds.$$

By Gronwall Lemma, we deduce that

$$\|\partial_t \tilde{u}(\cdot,t)\|^2_{L^2(\Omega)} + \|\tilde{u}(\cdot,t)\|^2_{H^1(\Omega)} + \int_0^t [(\partial_t \tilde{u}(a_-,s))^2 + (\partial_t \tilde{u}(a_+,s))^2]\, ds$$
$$\leq e^T\big(\|F\|^2_{L^2((0,T)\times\Omega)} + \|q\|^2_{L^2(\Omega)} + \|p\|^2_{H^1(\Omega)} + \sum_{\pm} \|H^\pm\|^2_{L^2(0,T)}\big),$$

which gives

$$\max_{|\alpha|\leq 1} \|\partial^\alpha \tilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t \tilde{u}(a_-,\cdot)\|^2_{L^2(0,T)} + \|\partial_t \tilde{u}(a_+,\cdot)\|^2_{L^2(0,T)}$$
$$\leq e^T\big(\|F\|^2_{L^2((0,T)\times\Omega)} + \|q\|^2_{L^2(\Omega)} + \|p\|^2_{H^1(\Omega)} + \sum_{\pm} \|H^\pm\|^2_{L^2(0,T)}\big). \quad (13)$$

Differentiating in time in (10), we now apply (13) to $\partial_t \tilde{u}$, and obtain:

$$\max_{|\alpha|\leq 1} \|\partial^\alpha \partial_t \tilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t^2 \tilde{u}(a_-,\cdot)\|^2_{L^2(0,T)} + \|\partial_t^2 \tilde{u}(a_+,\cdot)\|^2_{L^2(0,T)}$$
$$\leq e^T(\|\partial_t F\|^2_{L^2((0,T)\times\Omega)} + \|\partial_t^2 u(\cdot,0)\|^2_{L^2(\Omega)} + \|q\|^2_{H^1(\Omega)} + \sum_{\pm} \|\partial_t H^\pm\|^2_{L^2(0,T)}). \quad (14)$$

We must estimate $\partial_t^2 \tilde{u}(\cdot,0)$ in the righthand side of (14). We multiply (10) by $\partial_t^2 \tilde{u}$, integrate in space and evaluate at time 0:

$$\|\partial_t^2 \tilde{u}(\cdot,0)\|^2_{L^2(\Omega)} + (\partial_x p, \partial_x \partial_t^2 \tilde{u}(\cdot,0)) + (p, \partial_t^2 \tilde{u}(\cdot,0))$$
$$+ q(a_-)\partial_t^2 \tilde{u}(a_-,0) + q(a_+)\partial_t^2 \tilde{u}(a_+,0)$$
$$= H^-(0)\partial_t^2 \tilde{u}(a_-,0) + H^+(0)\partial_t^2 \tilde{u}(a_+,0) + (F(\cdot,0), \partial_t^2 \tilde{u}(\cdot,0)).$$

We integrate by parts in the second term, and rewrite the equality as

$$\|\partial_t^2 \tilde{u}(\cdot,0)\|^2_{L^2(\Omega)} - (\partial_x^2 \tilde{u}(\cdot,0), \partial_t^2 \tilde{u}(\cdot,0)) + (p, \partial_t^2 \tilde{u}(\cdot,0))$$
$$= (H^-(0) - q(a_-) + \partial_x p(a_-))\partial_t^2 \tilde{u}(a_-,0) + (H^+(0) - q(a_+) - \partial_x p(a_+))\partial_t^2 \tilde{u}(a_+,0) + (F(\cdot,0), \partial_t^2 \tilde{u}(\cdot,0)).$$

The boundary terms on the right-hand side vanish by the compatibility conditions, and we get

$$\|\partial_t^2 \tilde{u}(\cdot,0)\|^2_{L^2(\Omega)} = (\partial_x^2 p - p + F(\cdot,0), \partial_t^2 \tilde{u}(\cdot,0)).$$

Using the Cauchy-Schwarz Lemma, we obtain

$$\|\partial_t^2 \tilde{u}(\cdot,0)\|_{L^2(\Omega)} \leq \|\partial_x^2 p - p + F(\cdot,0)\|_{L^2(\Omega)}.$$

We replace the term $\|\partial_t^2 \tilde{u}(\cdot,0)\|_{L^2(\Omega)}$ in (13), and we deduce the second a priori estimate:

$$\begin{aligned}\max_{|\alpha|\leq 1} \|\partial^\alpha \partial_t \tilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t^2 \tilde{u}(a_-,\cdot)\|^2_{L^2(0,T)} + \|\partial_t^2 \tilde{u}(a_+,\cdot)\|^2_{L^2(0,T)} \\ \leq 3\, e^T \big(\|\partial_t F\|^2_{L^2((0,T)\times\Omega)} + \|F(\cdot,0)\|^2_{L^2(\Omega)} + \|p\|^2_{H^2(\Omega)} + \|q\|^2_{H^1(\Omega)} + \sum_{\pm} \|\partial_t H^\pm\|^2_{L^2(0,T)}\big). \quad (15)\end{aligned}$$

We still need to estimate the mixed derivatives $\partial_{xx}\tilde{u}$ in the interior and $\partial_{xt}\tilde{u}$ on the boundaries. We use the equation, which gives in the interior

$$\|\partial_{xx}\tilde{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq \|\partial_{tt}\tilde{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\tilde{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|F\|_{L^\infty(0,T;L^2(\Omega))}.$$

We now introduce the inequality

$$\|F\|^2_{L^\infty(0,T;L^2(\Omega))} \leq 2(\|F(\cdot,0)\|^2_{L^2(\Omega)} + T\|\partial_t F\|^2_{L^2(0,T;L^2(\Omega))}),$$

and by (15) and (13), we get, using that $e^T \geq 1$ and $e^T \geq T$,

$$\|\partial_{xx}\tilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))} \leq 15 e^T \big(\|F\|^2_{H^1(0,T;L^2(\Omega))} + \|F(\cdot,0)\|^2_{L^2(\Omega)} + \|p\|^2_{H^2(\Omega)} + \|q\|^2_{H^1(\Omega)} + \sum_{\pm} \|H^\pm\|^2_{H^1(0,T)}\big).$$

As for the boundary term, we get for instance on the left boundary

$$\|\partial_{tx}\tilde{u}(a_-,\cdot)\|_{L^2(0,T)} \leq \|\partial_{tt}\tilde{u}(a_-,\cdot)\|_{L^2(0,T)} + \|\partial_t H^-\|_{L^2(0,T)}.$$

Squaring the inequality, and adding the term coming from the right boundary leads to

$$\sum_{\pm} \|\partial_{tx}\tilde{u}(a_\pm,\cdot)\|^2_{L^2(0,T)} \leq 2\sum_{\pm}\big(\|\partial_{tt}\tilde{u}(a_\pm,\cdot)\|^2_{L^2(0,T)} + \|\partial_t H^\pm\|^2_{L^2(0,T)}\big),$$

which provides the last estimate announced in the proposition. The well-posedness is then derived in a standard way by the Galerkin method. ■

The solution $\bar{u}$ of the nonlinear subdomain problem is now defined through an iterative scheme. The initial guess is $\bar{u}_0 = p$. At step k, $\bar{u}_k$ being known, we define

$$\mathcal{F}(w) = w + f(w, \partial_t w, \partial_x w),\ \mathcal{G}^\pm(w) = g^\pm(w(a_\pm,\cdot)),\ \mathcal{H}^\pm(w) = h^\pm - \mathcal{G}^\pm(w). \quad (16)$$

$\bar{u}_{k+1}$ is the solution of the linear initial boundary value problem (10) with data $f_k = \mathcal{F}(\bar{u}_k)$, $g_k^\pm = \mathcal{G}^\pm(\bar{u}_k)$, $h_k^\pm = \mathcal{H}(\bar{u}_k)$, and initial data p and q. The proof of convergence for the sequence $\bar{u}_k$ is written in details in [11]. The uniqueness follows from the result:

Lemma 3.3 *There exists a real positive increasing function θ such that, for any time T, for any v in $V_2(\Omega,T)$, $\partial^\alpha\mathcal{F}(v)$ is in $V_1(\Omega,T)$, $\mathcal{G}^\pm(v)$ and $\mathcal{H}(v)$ are in $L^\infty(0,T)$. Moreover, for v_1, v_2 in $V_2(\Omega,T)$, we have*

$$\begin{aligned}\|\mathcal{H}^\pm(v_1) - \mathcal{H}^\pm(v_2)\|_{L^\infty(0,T)} &\leq \theta(\|v_1\|_{V_2(\Omega,T)} + \|v_2\|_{V_2(\Omega,T)})\|v_1 - v_2\|_{V_1(\Omega,T)}, \\ \|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{V_1(\Omega,T)} &\leq \theta(\|v_1\|_{V_2(\Omega,T)} + \|v_2\|_{V_2(\Omega,T)})\|v_1 - v_2\|_{V_2(\Omega,T)},\end{aligned} \quad (17)$$

As a consequence, we have the well-posedness of problem (4).

Corollary 3.4 *Let p in $H_0^2(\Omega)$ and q in $H_0^1(\Omega)$. There exists a time T_0^* such that for any $T \leq T_0^*$, (4) has a unique solution $\bar{u}$ in $V_2(\Omega,T)$, with $\partial^\alpha\bar{u}(a,\cdot)$ and $\partial^\alpha\bar{u}(b,\cdot)$ in $H^1(0,T)$ for $|\alpha| = 1$. Furthermore there exists a positive real number C^* depending only on the size of Ω such that*

$$\|\bar{u}\|^2_{V_2(\Omega,T)} + \sum_{|\alpha|=1} \|\partial^\alpha\bar{u}(a,\cdot)\|^2_{H^1(0,T)} \sum_{|\alpha|=1} \|\partial^\alpha\bar{u}(b,\cdot)\|^2_{H^1(0,T)} \leq C^*(\|p\|^2_{H^2(\Omega)} + \|q\|^2_{H^1(\Omega)}).$$

4 Convergence of The Algorithm

We now study the convergence of the Schwarz waveform Relaxation Algorithm (5). In order to define the algorithm, we need a regularity result:

Proposition 4.1 *For any ϵ, $0 < \epsilon < 1$, $V_2(\Omega, T) \subset \mathcal{C}^0(0, T; H^{2-\epsilon}(\Omega)) \cap \mathcal{C}^1(0, T; H^{1-\epsilon}(\Omega))$.*

Proof By using extension operators in time and space, it suffices to prove the result in $\mathbb{R} \times \mathbb{R}$. We make use of the Littlewood-Paley theory (see for example [2]). In particular, there exists φ and χ two tempered distributions on $\mathbb{R}$, with φ supported in $(-8/3, -3/4) \cup (3/4, 8/3)$, χ supported in $(-4/3, 4/3)$, and

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \ \forall \xi \in \mathbb{R}.$$

We define the dyadic projectors Δ_q by their action on a function u,

$$\Delta_{-1}u = \chi(D)u, \ \Delta_q u = \varphi(2^{-q}D)u \text{ for } q \geq 0, \tag{18}$$

where $D = -i\partial$. These operators give an equivalent norm in $H^s(\mathbb{R})$,

$$|u|_s = \left(\sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|^2 \right)^{\frac{1}{2}}.$$

They can also be used to define the Zygmund spaces

$$\mathcal{C}^r_* = \{u \in \mathcal{S}', \| u \|_r = \sup_{q \geq -1} 2^{qr} \|\Delta_q u\|^2 < +\infty\}.$$

$\mathcal{C}^r_*$ coïncides with the usual Hölder space when r is not an integer. For any positive r, we know that $W^{r,\infty}$, the space of functions in L^∞ with derivatives of order up to r in L^∞, is included in $\mathcal{C}^r_*$. Therefore we have

$$V_2(\mathbb{R}, \mathbb{R}) \subset \mathcal{C}^0_*(\mathbb{R}, H^2(\mathbb{R})) \cap \mathcal{C}^1_*(\mathbb{R}, H^1(\mathbb{R})) \cap \mathcal{C}^2_*(\mathbb{R}, L^2(\mathbb{R})).$$

We need an interpolation lemma.

Lemma 4.2 *For any positive α, β, a, b, for any θ, $0 \leq \theta \leq 1$,*

$$\mathcal{C}^\alpha_*(\mathbb{R}, H^a(\mathbb{R})) \cap \mathcal{C}^\beta_*(\mathbb{R}, H^b(\mathbb{R})) \subset \mathcal{C}^{\theta\alpha + (1-\theta)\beta}_*(\mathbb{R}, H^{\theta a + (1-\theta)b}(\mathbb{R})).$$

Applying the lemma with successively $(\alpha, \beta, a, b) = (0, 1, 2, 1)$ and $(\alpha, \beta, a, b) = (1, 2, 1, 0)$, we find for any θ, θ' in $(0, 1)$,

$$V_2(\mathbb{R}, \mathbb{R}) \subset \mathcal{C}^{1-\theta}_*(\mathbb{R}, H^{1+\theta}(\mathbb{R})) \cap \mathcal{C}^{2-\theta'}_*(\mathbb{R}, H^{\theta'}(\mathbb{R})).$$

Since for any $\epsilon > 0$ we have $\mathcal{C}^\epsilon_* \subset \mathcal{C}^0$ and $\mathcal{C}^{1+\epsilon}_* \subset \mathcal{C}^1$, this concludes the proof of Proposition 4.1.

Proof of Lemma 4.2 It relies on the convexity of the exponential function.

$$\|u\|^2_{\mathcal{C}^{\theta\alpha+(1-\theta)\beta}_*(\mathbb{R}, H^{\theta a+(1-\theta)b}(\mathbb{R}))} = \sup_{j \geq -1} 2^{2j(\theta\alpha + (1-\theta)\beta)} \sum_{k \geq -1} 2^{2k(\theta a + (1-\theta)b} \|\Delta^t_j \Delta^x_k u\|^2$$

where Δ^t_j (resp. Δ^x_j) is the Littlewood Paley operator acting in the time (resp. space) variable.

$$\begin{aligned} \sum_{k \geq -1} 2^{2k(\theta a + (1-\theta)b} \|\Delta^t_j \Delta^x_k u\|^2 &= \sum_{k \geq -1} \left(\left(2^{2ka} \|\Delta^t_j \Delta^x_k u\|^2 \right)^\theta \left(2^{2kb} \|\Delta^t_j \Delta^x_k u\|^2 \right)^{(1-\theta)} \right) \\ &\leq \left(\sum_{k \geq -1} 2^{2ka} \|\Delta^t_j \Delta^x_k u\|^2 \right)^\theta \left(\sum_{k \geq -1} 2^{2kb} \|\Delta^t_j \Delta^x_k u\|^2 \right)^{1-\theta}. \end{aligned}$$

Therefore we have

$$\|u\|^2_{\mathcal{C}_*^{\theta\alpha+(1-\theta)\beta}(\mathbb{R},H^{\theta a+(1-\theta)b}(\mathbb{R}))} \le \Big(\sup_{j\ge -1} 2^{2j\alpha}\sum_{k\ge -1} 2^{2ka}\|\Delta_j^t\Delta_k^x u\|^2\Big)^{\theta}\Big(\sup_{j\ge -1} 2^{2j\beta}\sum_{k\ge -1} 2^{2kb}\|\Delta_j^t\Delta_k^x u\|^2\Big)^{1-\theta}$$

which writes

$$\|u\|^2_{\mathcal{C}_*^{\theta\alpha+(1-\theta)\beta}(\mathbb{R},H^{\theta a+(1-\theta)b}(\mathbb{R}))} \le \Big(\|u\|^2_{\mathcal{C}_*^{\alpha}(\mathbb{R},H^{a}(\mathbb{R}))}\Big)^{\theta}\Big(\|u\|^2_{\mathcal{C}_*^{\beta}(\mathbb{R},H^{b}(\mathbb{R}))}\Big)^{1-\theta}.$$

■

Theorem 4.3 *Let p in $H_0^2(\Omega)$ and q in $H_0^1(\Omega)$. There exists a time $T_1 \le T_0^*$ such that for any $T \le T_1$, for any initial guess $h_i^{\pm}$ in $H^1(0,T)$ with the compatibility conditions $h_i^+(0) = q(a_{i+1})+p'(a_{i+1})+g^+(p(a_{i+1}))$ and $h_i^-(0) = q(a_i) - p'(a_i) + g^-(p(a_i))$, the algorithm (5) is defined and converges in $\cup_i V_2(\Omega_i,T)$ to the solution $\bar u$ of (4).*

Proof We first prove that the algorithm is well-defined : with the assumptions on $h_i^{\pm}$ in the theorem, we know by Theorem 3.1 that (5) defines in each Ω_i a u_i^1 in $V_2(\Omega_i,T)$, with $\partial_t u_i^1(a_i,\cdot)$, $\partial_t u_i^1(a_{i+1},\cdot)$, $\partial_x u_i^1(a_i,\cdot)$ and $\partial_x u_i^1(a_{i+1},\cdot)$ in $H^1(0,T)$ for $T\le T_i^k$. Furthermore, by Lemma 3.3, $\mathcal{B}^-(g^-)u_{i-1}^1(a_i,\cdot)$ and $\mathcal{B}^+(g^+)u_{i+1}^1(a_{i+1},\cdot)$ are in $H^1(0,T)$. As for the compatibility conditions, we have

$$\mathcal{B}^-(g^-)u_{i-1}^1(a_i,0) = \lim_{t\to 0}\big(\partial_t u_{i-1}^1(a_i,t) - \partial_x u_{i-1}^1(a_i,t) + g^-(u_{i-1}^1(a_i,t))\big)$$

and by Proposition 4.1, we can pass to the limit and get

$$\mathcal{B}^-(g^-)u_{i-1}^1(a_i,0) = q(a_i) - p'(a_i) + g^-(p(a_i)).$$

This, together with the same regularity result on a_{i+1}, permits the recursion.

We define for $T\le \min(T_0^*, \min_i(T_i^{*k}))$, for $k\ge 1$, the quantities (with $u_i = \bar u/_{\Omega_i}$) for $1\le i,j\le I$, $j=i$ or $j=i-1$,

$$\begin{aligned} e_i^k &= u_i^k - u_i,\\ f_i^k &= \mathcal{F}(u_i^k) - \mathcal{F}(u_i)\\ h_{i,j}^{k,-} &= g^-(u_j^k(a_i)) - g^-(u_j(a_i)), \quad h_{i,j}^{k,+} = g^+(u_j^k(a_{i+1})) - g^+(u_j(a_{i+1})). \end{aligned}$$

The operators $\mathcal{F}$ is defined in (16). The error e_i^k in Ω_i at iteration k is a solution of

$$(\partial_t^2 - \partial_x^2)e_i^k + e_i^k = f_i^k \text{ in } \Omega_i\times(0,T) \tag{19}$$

$$(\partial_t - \partial_x)e_i^k + h_{i,i}^{k,-} = (\partial_t - \partial_x)e_{i-1}^{k-1} + h_{i,i-1}^{k-1,-} \text{ on } \{a_i\}\times(0,T) \tag{20}$$

$$(\partial_t + \partial_x)e_i^k + h_{i,i}^{k,+} = (\partial_t + \partial_x)e_{i+1}^{k-1} + h_{i,i+1}^{k-1,+} \text{ on } \{a_{i+1}\}\times(0,T) \tag{21}$$

with vanishing initial values and $e_0^k\equiv 0$, $e_{I+2}^k\equiv 0$, $h_{1,0}^{k,-}\equiv 0$, $h_{I,I+1}^{k,+}=0$. In order to get a new energy estimate in Ω_i, we multiply (19) by $\partial_t e_i^k$ and integrate by parts:

$$\frac{d}{dt}E_{\Omega_i}(e_i^k) - [\partial_t e_i^k\partial_x e_i^k(a_{i+1},\cdot) - \partial_t e_i^k\partial_x e_i^k(a_i,\cdot)] = (f_i^k, \partial_t e_i^k) \tag{22}$$

with $E_\Omega(u) = \frac12(\|\partial_t u\|^2_{L^2(\Omega)} + \|\partial_x u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})$. We rewrite the boundary terms using the boundary operators:

$$\begin{aligned} \partial_t e_i^k\partial_x e_i^k(a_{i+1},\cdot) &= \frac14((\partial_t+\partial_x)e_i^k(a_{i+1},\cdot) + h_{i,i}^{k,+})^2 - \frac14((\partial_t-\partial_x)e_i^k(a_{i+1},\cdot) + h_{i+1,i}^{k,-})^2 + R_{i,i+1}^k,\\ -\partial_t e_i^k\partial_x e_i^k(a_i,\cdot) &= \frac14((\partial_t-\partial_x)e_i^k(a_i,\cdot) + h_{i,i}^{k,-})^2 - \frac14((\partial_t+\partial_x)e_i^k(a_i,\cdot) + h_{i-1,i}^{k,+})^2 + R_{i,i-1}^k. \end{aligned} \tag{23}$$

The remainders $R^k_{i,i+1}$ and $R^k_{i,i-1}$ will be evaluated later. We insert (23) into (22), and obtain

$$\frac{d}{dt}E_{\Omega_i}(e_i^k) + \frac{1}{4}((\partial_t - \partial_x)e_i^k(a_{i+1},\cdot) + h^{k,-}_{i+1,i})^2 + \frac{1}{4}((\partial_t + \partial_x)e_i^k(a_i,\cdot) + h^{k,+}_{i-1,i})^2$$
$$= \frac{1}{4}((\partial_t + \partial_x)e_i^k(a_{i+1},\cdot) + h^{k,+}_{i,i})^2 + \frac{1}{4}((\partial_t - \partial_x)e_i^k(a_i,\cdot) + h^{k,-}_{i,i})^2 + R^k_{i,i+1} + R^k_{i,i-1} + (f_i^k, \partial_t e_i^k).$$

Using the transmission conditions (20), (21), we get

$$\begin{aligned}\frac{d}{dt}E_{\Omega_i}(e_i^k) &+ \frac{1}{4}((\partial_t - \partial_x)e_i^k(a_{i+1},\cdot) + h^{k,-}_{i+1,i})^2 + \frac{1}{4}((\partial_t + \partial_x)e_i^k(a_i,\cdot) + h^{k,+}_{i-1,i})^2\\ &= \frac{1}{4}((\partial_t + \partial_x)e_{i+1}^{k-1}(a_{i+1},\cdot) + h^{k-1,+}_{i,i+1})^2 + \frac{1}{4}((\partial_t - \partial_x)e_{i-1}^{k-1}(a_i,\cdot) + h^{k-1,-}_{i,i-1})^2\\ &+ R^k_{i,i+1} + R^k_{i,i-1} + (f_i^k, \partial_t e_i^k).\end{aligned} \tag{24}$$

We sum (24) on the indexes i, $1 \leq i \leq I$ and integrate in time. We translate the domain indexes in the right-hand side. Defining

$$E_{\partial\Omega_i}(e_i^k) = \frac{1}{4}[((\partial_t - \partial_x)e_i^k(a_{i+1},\cdot) + h^{k,-}_{i+1,i})^2 + ((\partial_t + \partial_x)e_i^k(a_i,\cdot) + h^{k,+}_{i-1,i})^2],$$

we get, since the initial data vanish,

$$\begin{aligned}\sum_{i=1}^{I} E_{\Omega_i}(e_i^k)(t) \quad &+ \int_0^t \sum_{i=1}^{I} E_{\partial\Omega_i}(e_i^k)(s)ds \leq \int_0^t \sum_{i=1}^{I} E_{\partial\Omega_i}(e_i^{k-1})(s)ds\\ &+ \int_0^t \sum_{i=1}^{I}(R^k_{i,i+1} + R^k_{i,i-1})(s)ds + \int_0^t \sum_{i=1}^{I}(f_i^k, \partial_t e_i^k)(s)ds.\end{aligned} \tag{25}$$

Differentiating the equation and the transmission conditions in time yields the bound on $\partial_t e_i^k$:

$$\begin{aligned}\sum_{i=1}^{I} E_{\Omega_i}(\partial_t e_i^k)(t) \quad &+ \int_0^t \sum_{i=1}^{I} E_{\partial\Omega_i}(\partial_t e_i^k)(s)ds \leq \int_0^t \sum_{i=1}^{I} E_{\partial\Omega_i}(\partial_t e_i^{k-1})(s)ds\\ &+ \int_0^t \sum_{i=1}^{I}(\tilde{R}^k_{i,i+1} + \tilde{R}^k_{i,i-1})(s)ds + \int_0^t \sum_{i=1}^{I}(\partial_t f_i^{k+1}, \partial_{tt} e_i^k)(s)ds.\end{aligned} \tag{26}$$

We now estimate the remainders. We start with $R^k_{i,i+1}$ (ignoring the superscript k):

$$R_{i,i+1} = \frac{1}{4}[-h^+_{i,i}(2(\partial_t e_i + \partial_x e_i)(a_{i+1},\cdot) + h^+_{i,i}) + h^-_{i+1,i}(2(\partial_t e_i - \partial_x e_i)(a_{i+1},\cdot) + h^-_{i+1,i})],$$

and we get a bound on the integral of $R_{i,i+1}$:

$$\int_0^t R_{i,i+1}(s)ds \leq \frac{3}{4}(\|h^+_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|h^-_{i+1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) + \frac{1}{2}(\|\partial_t e_i(a_{i+1},\cdot)\|^2_{H^{-\frac{1}{2}}(0,t)} + \|\partial_x e_i(a_{i+1},\cdot)\|^2_{H^{-\frac{1}{2}}(0,t)}).$$

We can treat $R_{i,i-1}$,$\tilde{R}_{i,i-1}$, and $\tilde{R}_{i,i-1}$ the same way and obtain

$$\begin{aligned}\int_0^t R_{i,i-1}(s)ds \leq \quad & \tfrac{3}{4}(\|h^-_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|h^+_{i-1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) + \tfrac{1}{2}(\|\partial_t e_i(a_i,\cdot)\|^2_{H^{-\frac{1}{2}}(0,t)} + \|\partial_x e_i(a_i,\cdot)\|^2_{H^{-\frac{1}{2}}(0,t)}),\\ \int_0^t \tilde{R}_{i,i+1}(s)ds \leq \quad & \tfrac{3}{4}(\|\partial_t h^+_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^-_{i+1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) + \tfrac{1}{2}(\|\partial_t^2 e_i(a_{i+1},\cdot)\|^2_{H^{-\frac{1}{2}}(0,t)} + \|\partial_{xt} e_i(a_{i+1},\cdot)\|^2_{H^{-\frac{1}{2}}(0,t)}),\\ \int_0^t \tilde{R}_{i,i-1}(s)ds \leq \quad & \tfrac{3}{4}(\|\partial_t h^-_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^+_{i-1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) + \tfrac{1}{2}(\|\partial_t^2 e_i(a_i,\cdot)\|^2_{H^{-\frac{1}{2}}(0,t)} + \|\partial_{xt} e_i(a_i,\cdot)\|^2_{H^{-\frac{1}{2}}(0,t)}).\end{aligned}$$

At point a_i for instance, by the Trace Theorem, there is a constant C_3 independent of T, such that for any α with $|\alpha| = 1$, we have

$$\|\partial^\alpha e_i(a_i,\cdot)\|_{H^{-\frac{1}{2}}(0,t)} \le \|\partial^\alpha e_i(a_i,\cdot)\|_{L^2(0,t)} \le C_3 \|\partial^\alpha e_i\|_{H^1(\Omega_i\times(0,t))},$$
$$\|\partial^\alpha \partial_t e_i(a_i,\cdot)\|_{H^{-\frac{1}{2}}(0,t)} \le \|\partial^\alpha e_i(a_i,\cdot)\|_{H^{\frac{1}{2}}(0,t)} \le C_3 \|\partial^\alpha e_i\|_{H^1(\Omega_i\times(0,t))},$$

which gives our first bounds on the remainders:

$$\int_0^t R_{i,i+1}(s)ds \le \frac{1}{2}(\|h^+_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|h^-_{i+1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) + \frac{C_3^2}{2}\|e_i\|^2_{H^2(\Omega_i\times(0,t))},$$
$$\int_0^t \tilde{R}_{i,i+1}(s)ds \le \frac{1}{2}(\|\partial_t h^+_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^-_{i+1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) + \frac{C_3^2}{2}\|e_i\|^2_{H^2(\Omega_i\times(0,t))}.$$

We now insert the previous estimates in (25) and (26). By Cauchy-Schwarz inequality we get

$$\begin{aligned}\sum_{i=1}^I E_{\Omega_i}(e_i^k)(t) + \int_0^t \sum_{i=1}^I E_{\partial\Omega_i}(e_i^k)(s)ds \le \\ \int_0^t \sum_{i=1}^I E_{\partial\Omega_i}(e_i^{k-1})(s)ds + \frac{C_3^2+1}{2}\sum_{i=1}^I \|e_i^k\|^2_{H^2(\Omega_i\times(0,t))} + \frac{1}{2}\sum_{i=1}^I\int_0^t \|f_i^k\|^2_{L^2(\Omega_i)}(s)ds \\ + \frac{3}{4}\sum_{i=1}^I(\|h^{k,+}_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|h^{k,-}_{i+1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) + \frac{3}{4}\sum_{i=1}^I(\|h^{k,-}_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|h^{k,+}_{i-1,i}\|^2_{H^{\frac{1}{2}}(0,t)}),\end{aligned} \tag{27}$$

$$\begin{aligned}\sum_{i=1}^I E_{\Omega_i}(\partial_t e_i^k)(t) + \int_0^t \sum_{i=1}^I E_{\partial\Omega_i}(\partial_t e_i^k)(s)ds \le \\ \int_0^t \sum_{i=1}^I E_{\partial\Omega_i}(\partial_t e_i^{k-1})(s)ds + \frac{C_3^2+1}{2}\sum_{i=1}^I \|e_i^k\|^2_{H^2(\Omega_i\times(0,t))} + \frac{1}{2}\sum_{i=1}^I\int_0^t \|\partial_t f_i^k\|^2_{L^2(\Omega_i)}(s)ds \\ + \frac{3}{4}\sum_{i=1}^I(\|\partial_t h^{k,+}_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^{k,-}_{i+1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) + \frac{3}{4}\sum_{i=1}^I(\|\partial_t h^{k,-}_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^{k,+}_{i-1,i}\|^2_{H^{\frac{1}{2}}(0,t)})\end{aligned} \tag{28}$$

Adding (27) and (28), we can write

$$\begin{aligned}&\sum_{i=1}^I(E_{\Omega_i}(e_i^k) + E_{\Omega_i}(\partial_t e_i^k))(t) + \int_0^t\sum_{i=1}^I(E_{\partial\Omega_i}(e_i^k) + E_{\partial\Omega_i}(\partial_t e_i^k))(s)ds \\ &\le \int_0^t \sum_{i=1}^I(E_{\partial\Omega_i}(e_i^{k-1}) + E_{\partial\Omega_i}(\partial_t e_i^{k-1}))(s)ds + (C_3^2+1)\sum_{i=1}^I\|e_i^k\|^2_{H^2(\Omega_i\times(0,t))} \\ &+\frac{3}{4}\sum_{i=1}^I(\|h^{k,+}_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|h^{k,+}_{i+1,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|h^{k,-}_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|h^{k,+}_{i-1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) \\ &+\frac{3}{4}\sum_{i=1}^I(\|\partial_t h^{k,+}_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^{k,+}_{i+1,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^{k,-}_{i,i}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^{k,+}_{i-1,i}\|^2_{H^{\frac{1}{2}}(0,t)}) \\ &+\sum_{i=1}^I\frac{1}{2}(\|f_i^k\|^2_{L^2(\Omega\times(0,T))} + \|\partial_t f_i^k\|^2_{L^2(\Omega\times(0,T))}).\end{aligned}$$

We now estimate the quantities involving the $h^{k,\pm}_{i,j}$. Refining the results in Lemma 3.3, we have a real positive increasing function θ_2, such that

$$\|h^{k,\pm}_{i,j}\|^2_{H^{\frac{1}{2}}(0,t)} + \|\partial_t h^{k,\pm}_{i,j}\|^2_{H^{\frac{1}{2}}(0,t)} \le \theta_2^2(\sum_{\substack{|\alpha|\le 2\\ \alpha\ne(0,2)}}(\|\partial^\alpha u_i\|^2_{L^2(\Omega\times(0,T))} + \|\partial^\alpha u_i^k\|^2_{L^2(\Omega\times(0,T))}))\sum_{\substack{|\alpha|\le 2\\ \alpha\ne(0,2)}}\|\partial^\alpha e_i^k\|^2_{L^2(\Omega\times(0,T))}$$

which gives

$$\begin{aligned}\sum_{i=1}^{I}(E_{\Omega_i}(e_i^k) + E_{\Omega_i}(\partial_t e_i^k))(t) + \int_0^t \sum_{i=1}^{I}(E_{\partial\Omega_i}(e_i^k) + E_{\partial\Omega_i}(\partial_t e_i^k))(s)ds \\ \leq \int_0^t \sum_{i=1}^{I}(E_{\partial\Omega_i}(e_i^{k-1}) + E_{\partial\Omega_i}(\partial_t e_i^{k-1}))(s)ds \\ + \sum_{i=1}^{I}\theta_3(\sum_{\substack{|\alpha|\leq 2\\ \alpha\neq(0,2)}}(\|\partial^\alpha u_i\|^2_{L^2(\Omega\times(0,T))} + \|\partial^\alpha u_i^k\|^2_{L^2(\Omega\times(0,T))}))\|e_i^k\|^2_{H^2(\Omega_i\times(0,t))} \\ + \sum_{i=1}^{I}\frac{1}{2}(\|f_i^k\|^2_{L^2(\Omega\times(0,T))} + \|\partial_t f_i^k\|^2_{L^2(\Omega\times(0,T))}),\end{aligned} \tag{29}$$

with $\theta_3 = 3\theta_2^2 + C_3^2 + 1$. We now evaluate the terms in the right-hand side. We first note that

$$\|e_i^k\|^2_{H^2(\Omega_i\times(0,t))} \leq \int_0^t (E_{\Omega_i}(e_i^k) + E_{\Omega_i}(\partial_t e_i^k) + \|\partial_{xx} e_i^k\|^2_{L^2(\Omega_i)})ds,$$

and evaluate $\|\partial_{xx} e_i^k\|^2_{L^2(\Omega_i\times(0,t))}$ by equation (19):

$$\|\partial_{xx} e_i^k\|^2_{L^2(\Omega_i\times(0,t))} \leq 3(\|\partial_{tt} e_i^k\|^2_{L^2(\Omega_i\times(0,t))} + \|e_i^k\|^2_{L^2(\Omega_i\times(0,t))} + \|f_i^k\|^2_{L^2(\Omega_i\times(0,t))}),$$

from which we deduce

$$\|e_i^k\|^2_{H^2(\Omega_i\times(0,t))} \leq 4\int_0^t (E_{\Omega_i}(e_i^k) + E_{\Omega_i}(\partial_t e_i^k))ds + 3\|f_i^k\|^2_{L^2(\Omega_i\times(0,t))}.$$

There remains only in (29)

$$\begin{aligned}\sum_{i=1}^{I}(E_{\Omega_i}(e_i^k) + E_{\Omega_i}(\partial_t e_i^k))(t) + \int_0^t \sum_{i=1}^{I}(E_{\partial\Omega_i}(e_i^k) + E_{\partial\Omega_i}(\partial_t e_i^k))(s)ds \\ \leq \int_0^t \sum_{i=1}^{I}(E_{\partial\Omega_i}(e_i^{k-1}) + E_{\partial\Omega_i}(\partial_t e_i^{k-1}))(s)ds \\ + \sum_{i=1}^{I}4\theta_3(\sum_{\substack{|\alpha|\leq 2\\ \alpha\neq(0,2)}}(\|\partial^\alpha u_i\|^2_{L^2(\Omega\times(0,T))} + \|\partial^\alpha u_i^k\|^2_{L^2(\Omega\times(0,T))}))\int_0^t (E_{\Omega_i}(e_i^k) + E_{\Omega_i}(\partial_t e_i^k))ds \\ + \sum_{i=1}^{I}(3\theta_3(\sum_{\substack{|\alpha|\leq 2\\ \alpha\neq(0,2)}}(\|\partial^\alpha u_i\|^2_{L^2(\Omega\times(0,T))} + \|\partial^\alpha u_i^k\|^2_{L^2(\Omega\times(0,T))})) + \frac{1}{2})(\|f_i^k\|^2_{L^2(\Omega\times(0,T))} + \|\partial_t f_i^k\|^2_{L^2(\Omega\times(0,T))}).\end{aligned} \tag{30}$$

Again, as in Lemma 3.3, there exists a positive increasing function θ_4 such that

$$\|f_i^k(t,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t f_i^k(t,\cdot)\|^2_{L^2(\Omega)} \leq \theta_4^2(\sum_{\substack{|\alpha|\leq 2\\ \alpha\neq(0,2)}}(\|\partial^\alpha u_i\|^2_{L^2(\Omega)} + \|\partial^\alpha u_i^k\|^2_{L^2(\Omega)})\sum_{\substack{|\alpha|\leq 2\\ \alpha\neq(0,2)}}\|\partial^\alpha e_i^k\|^2_{L^2(\Omega)},$$

the latter sum means that no term ∂_{xx} are present, therefore we can bound the sum by twice the energy. Furthermore we know that the energy of $\bar{u}$ is bounded on the interval $(0,T)$. Thus there exists a new positive increasing function θ_5, depending on u, such that

$$\|f_i^k(t,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t f_i^k(t,\cdot)\|^2_{L^2(\Omega)} \leq (\theta_5^2(E_{\Omega_i}(e_i^k) + E_{\Omega_i}(\partial_t e_i^k))(E_{\Omega_i}(e_i^k) + E_{\Omega_i}(\partial_t e_i^k)))(t). \tag{31}$$

We insert (31) into (30), and get (with a new function θ_6)

$$\begin{aligned}\sum_{i=1}^{I}(E_{\Omega_i}(e_i^k)+E_{\Omega_i}(\partial_t e_i^k))(t)+\int_0^t\sum_{i=1}^{I}(E_{\partial\Omega_i}(e_i^k)+E_{\partial\Omega_i}(\partial_t e_i^k))(s)ds\\ \leq \int_0^t\sum_{i=1}^{I}(E_{\partial\Omega_i}(e_i^{k-1})+E_{\partial\Omega_i}(\partial_t e_i^{k-1}))(s)ds\\ +\sum_{i=1}^{I}\int_0^t(\theta_6(E_{\Omega_i}(e_i^k)+E_{\Omega_i}(\partial_t e_i^k)))(E_{\Omega_i}(e_i^k)+E_{\Omega_i}(\partial_t e_i^k))(s)ds.\end{aligned}\tag{32}$$

For clarity we define

$$E_{int}^k=\sum_{i=1}^{I}(E_{\Omega_i}(e_i^k)+E_{\Omega_i}(\partial_t e_i^k)),\quad E_b^k=\sum_{i=1}^{I}(E_{\partial\Omega_i}(e_i^k)+E_{\partial\Omega_i}(\partial_t e_i^k)).$$

and we can rewrite (32) as

$$E_{int}^k(t)+\int_0^t E_b^k(s)ds\leq\int_0^t E_b^{k-1}(s)ds+\int_0^t\theta_6(E_{int}^k(s))E_{int}^k(s)ds \tag{33}$$

Summing in k, we define $\widetilde{E}_{int}^K=\sum_{k=1}^K E_{int}^k$, and we have

$$\widetilde{E}_{int}^K(t)+\int_0^t E_b^K(s)ds\leq\int_0^t E_b^0(s)ds+\int_0^t\theta_6(\widetilde{E}_{int}^K(s))\widetilde{E}_{int}^K(s)ds.$$

Let now $C>0$. If t tends to 0, $\int_0^t E_b^1(s)ds+tC\theta_6(C)$ tends to 0. Therefore there exists a T_1 such that $\int_0^{T_1}E_b^1(s)ds+T_1C\theta_6(C)=C$, and so we have for $t\leq T_1$,

$$\widetilde{E}_{int}^K(t)\leq C.$$

We conclude that u_i^k exists on the time interval $(0,T_1)$, and that $\sum_{i=1}^I(E_{\Omega_i}(e_i^k)+E_{\Omega_i}(\partial_t e_i^k))$ tends to 0 when k tends to infinity: the sequence u_i^k converges to $\bar{u}$ on $(0,T_1)$ in each subdomain in the norm of energy. ■

5 A Finite Volume Discretization

We use here a finite volumes scheme, which has been described in [7] for the linear one-dimensional wave equation, and extended to the non linear boundary value problems in the frame of absorbing boundary conditions in [10]. We restrict ourselves to uniform meshes in time and space.

5.1 Discretization of the Subdomain Problem (6)

The domain $\Omega\times(0,T)$ is meshed by a rectangular grid, with uniform mesh sizes Δx and Δt. There are $J+1$ points in space with $\Delta x=(a_+-a_-)/J$, and $N+1$ points in time, with $\Delta t=T/N$. We denote the numerical approximation to $u(a_-+j\Delta x,n\Delta t)$ by $U(j,n)$. We introduce the notations:

$$\begin{aligned}D_t^+U(j,n)&=\frac{U(j,n+1)-U(j,n)}{\Delta t}, & D_t^-U(j,n)&=\frac{U(j,n)-U(j,n-1)}{\Delta t},\\ D_x^+U(j,n)&=\frac{U(j+1,n)-U(j,n)}{\Delta x}, & D_x^-U(j,n)&=\frac{U(j,n)-U(j-1,n)}{\Delta x},\\ D_x^0U(j,n)&=\frac{U(j+1,n)-U(j-1,n)}{2\Delta x}, & D_t^0U(j,n)&=\frac{U(j,n+1)-U(j,n-1)}{2\Delta t},\\ D_t^{--}U(j,n)&=\frac{3U(j,n)-4U(j,n-1)+U(j,n-2)}{2\Delta t}, & D_t^{-*}U(j,n)&=\begin{cases}D_t^{--}U(j,n)\text{ for }n\geq 2,\\ D_t^-U(j,n)\text{ for }n=1.\end{cases}\end{aligned}\tag{34}$$

The last finite derivative in (34) is a second order approximation of $\partial_t u$, to be used in the nonlinear term, in order to design an explicit scheme.

The scheme in the interior writes

$$\left(D_t^+ D_t^- - D_x^+ D_x^-\right) U(j,n) - f(U(j,n), D_t^{-*} U(j,n), D_x^0 U(j,n)) = 0 \quad \text{in } [\![1, J-1]\!] \times [\![1, N-1]\!]. \tag{35}$$

We define the discrete initial value as

$$P(j) = p(a_- + j\Delta x), \quad Q(j) = q(a_- + j\Delta x),$$

and we obtain the initial scheme

$$(D_t^+ - \frac{\Delta t}{2} D_x^+ D_x^-) U(j,0) = Q(j) + \frac{\Delta t}{2} f(P(j), Q(j), D_x^0 Q(j)), \quad \text{in } [\![1, J-1]\!]. \tag{36}$$

For the boundary conditions, we define the discrete boundary operators as:

$$\begin{aligned}
B^-(f,g^-)\, U(0,n) &= (D_t^0 - D_x^+ + \frac{\Delta x}{2} D_t^+ D_t^-) U(0,n) - \frac{\Delta x}{2} f(U(0,n), D_t^{-*} U(0,n), D_x^+ U(0,n)) + g^-(U(0,n)),\\
B^-(f,g^-)\, U(0,0) &= (D_t^+ - D_x^+ + \frac{\Delta x}{\Delta t} D_t^+)\, U(0,0) - \frac{\Delta x}{\Delta t} Q(0) - \frac{\Delta x}{2} f(P(0), Q_i(0), D_x^+ P(0)) + g^-(P(0)),
\end{aligned} \tag{37}$$

$$\begin{aligned}
B^+(f,g^+)\, U(J,n) &= (D_t^0 + D_x^- + \frac{\Delta x}{2} D_t^+ D_t^-) U(J,n) - \frac{\Delta x}{2} f(U(J,n), D_t^{-*} U_i(J,n), D_x^- U(J,n)) + g^+(U(J,n)).\\
B^+(f,g^+)\, U(J,0) &= (D_t^+ + D_x^- + \frac{\Delta x}{\Delta t} D_t^+) U(J,0) - \frac{\Delta x}{\Delta t} Q(J) - \frac{\Delta x}{2} f(P(J), Q_i(J), D_x^- P(J)) + g^+(P(J)).
\end{aligned} \tag{38}$$

We define the boundary data as

$$H^\pm(n) = \frac{1}{\Delta t} \int_{t_n - \Delta t/2}^{t_n + \Delta t/2} h^\pm(\tau) d\tau, \quad 1 \le n \le N, H^\pm(0) = \frac{2}{\Delta t} \int_0^{\Delta t/2} h^\pm(\tau) d\tau \tag{39}$$

The discretization of Problem (6) is now given by (35), (36), with boundary conditions

$$B^-(f,g^-) U(0,\cdot) = H^-\ , \ B^+(f,g^+) U(J,\cdot) = H^+ \text{ in } [\![0, n \le N]\!], \tag{40}$$

where the discrete boundary operators $B^\pm$ are given in (37), (38).

Our numerical computations indicate that this scheme is second order both in space and time.

5.2 The Discrete Schwarz Waveform Relaxation Algorithm

The equation is now discretized on each subdomain $\Omega_i \times (0,T)$, $i = 1, \dots, I$ separately, using an uniform mesh with sizes Δx and Δt. There are $J_i + 1$ points in space and $N+1$ grid points in time in subdomain Ω_i, with $\Delta x = (a_{i+1} - a_i)/J_i$ and $\Delta t = T/N$. We denote the numerical approximation to $u_i^k(a_i + j\Delta x, n\Delta t)$ on Ω_i at iteration step k by $U_i^k(j,n)$.

The problem in domain Ω_i is now defined through boundary data $H_i^{\pm, k-1}$, coming from the neighboring subdomains $\Omega_{i\pm1}$ .Therefore, we define the extraction operator from domain Ω_i to his neighbours as:

$$\begin{aligned}
\widetilde{B}^+(f,g^+)\, U_i(0,n) &= (D_t^0 + D_x^+ - \frac{\Delta x}{2} D_t^+ D_t^-) U_i(0,n) + \frac{\Delta x}{2} f(U_i(0,n), D_t^{-*} U_i(0,n), D_x^+ U_i(0,n)) + g^+(U_i(0,n)),\\
\widetilde{B}^+(f,g^+)\, U_i(0,0) &= (D_t^+ + D_x^+ - \frac{\Delta x}{\Delta t} D_t^+)\, U_i(0,0) + \frac{\Delta x}{\Delta t} Q_i(0) + \frac{\Delta x}{2} f(P_i(0), Q_i(0), D_x^+ P_i(0)) + g^+(P_i(0)).
\end{aligned} \tag{41}$$

$$\begin{aligned}
\widetilde{B}^-(f,g^-) U_i(J_i,n) &= (D_t^0 - D_x^- - \frac{\Delta x}{2} D_t^+ D_t^-) U_i(J_i,n) + \frac{\Delta x}{2} f(U_i(J_i,n), D_t^{-*} U_i(J_i,n), D_x^- U_i(J_i,n)) + g^-(U_i(J_i,n)),\\
\widetilde{B}^-(f,g^-)\, U_i(J_i,0) &= (D_t^+ - D_x^- - \frac{\Delta x}{\Delta t} D_t^+)\, U_i(J_i,0) + \frac{\Delta x}{\Delta t} Q_i(J_i) + \frac{\Delta x}{2} f(P_i(J_i), Q_i(J_i), D_x^- P_i(J_i)) + g^-(P_i(J_i)).
\end{aligned} \tag{42}$$

The discrete Schwarz waveform relaxation algorithm on subdomains Ω_i, $i = 1, \ldots, I$ is defined as follows. An initial guess $\{H_i^{0,\pm}\}_{1\le i\le I}$ is given. For $k \ge 1$, we solve

$$\left[\begin{array}{l} \left(D_t^+ D_t^- - D_x^+ D_x^-\right) U_i^k = f(U_i^k, D_t^{-*} U_i^k, D_x^0 U_i^k) \text{ in } [\![1, J_i - 1]\!] \times [\![1, N]\!], \\ U_i^k(\cdot, 0) = P_i, \quad (D_t^+ - \frac{\Delta t}{2} D_x^+ D_x^-) U_i^k(\cdot, 0) = Q_i + \frac{\Delta t}{2} f(P_i, Q_i, D_x^0 Q_i) \text{ in } [\![1, J_i - 1]\!], \\ B^-(f, g^-) U_i^k(0, \cdot) = H_i^{-,k-1}, \; B^+(f, g^+) U_i^k(J_i, \cdot) = H_i^{+,k-1} \text{ in } [\![0, n \le N]\!] \end{array}\right. \tag{43}$$
$$H_i^{-,k} = \widetilde{B}^-(f, g^-) U_{i-1}^k(J_{i-1}, \cdot), \; H_i^{+,k} = \widetilde{B}^+(f, g^+) U_{i+1}^k(0, \cdot) \text{ in } [\![0, n \le N]\!].$$

As in the continuous algorithm, we set $U_0^k \equiv 0$ and $U_{I+2}^k \equiv 0$.

We denote by $\bar{U}$ the discrete approximation of problem (4), obtained by solving (35,36,40) on $\Omega = (a, b)$ with $J = \sum_{i=1}^{I} J_i$ intervals of length Δx, and $H^\pm = 0$. Each subproblem is an explicit scheme, thus has a unique solution. Therefore the Schwarz waveform Relaxation Algorithm is well defined. If it converges, the limit in each subdomain is denoted by V_i. It satisfies the same scheme as $\bar{U}$ at initial time, in the interior and on the exterior boundaries. At point a_i it satisfies for any $n \ge 0$

$$B^-(f, g^-) V_i(0, n) = \widetilde{B}^-(f, g^-) V_{i-1}(J_{i-1}, n), \; B^+(f, g^+) V_{i-1}(J_{i-1}, n) = \widetilde{B}^+(f, g^+) V_i(0, n). \tag{44}$$

Theorem 5.1 *Suppose that f is affine in the third variable $\partial_x u$. Then, if the discrete algorithm converges, it converges to the discrete approximation $\bar{U}$ of problem (4).*

Proof Since p is in $H^2(\Omega)$ and q is in $H^1(\Omega)$, there are both continuous and we have for any i $V_i(0,0) = P_i(0) = V_{i-1}(J_{i-1}, 0) = P_{i-1}(J_{i-1}) = p(a_i)$, and $Q_{i-1}(J_{i-1}) = Q_i(0) = q(a_i)$. We write the transmission conditions (44) for $n = 0$. The nonlinear terms containing $g^\pm$ on both sides cancel out, and we have

$$\begin{aligned} &(D_t^+ - D_x^+ + \frac{\Delta x}{\Delta t} D_t^+) V_i(0, 0) - \frac{\Delta x}{\Delta t} q(a_i) - \frac{\Delta x}{2} f(p(a_i), q(a_i), D_x^+ p(a_i)) = \\ &\qquad (D_t^+ - D_x^- - \frac{\Delta x}{\Delta t} D_t^+) V_{i-1}(J_{i-1}, 0) + \frac{\Delta x}{\Delta t} q(a_i) + \frac{\Delta x}{2} f(p(a_i), q(a_i), D_x^- p(a_i)), \end{aligned} \tag{45}$$

$$\begin{aligned} &(D_t^+ + D_x^+ - \frac{\Delta x}{\Delta t} D_t^+) V_i(0, 0) + \frac{\Delta x}{\Delta t} q(a_i) + \frac{\Delta x}{2} f(p(a_i), q(a_i), D_x^+ p(a_i)) = \\ &\qquad (D_t^+ + D_x^- + \frac{\Delta x}{\Delta t} D_t^+) V_{i-1}(J_{i-1}, 0) - \frac{\Delta x}{\Delta t} q(a_i) - \frac{\Delta x}{2} f(p(a_i), q(a_i)), D_x^- p(a_i)). \end{aligned} \tag{46}$$

Adding (45) and (46) yields $D_t^+ V_i(0, 0) = D_t^+ V_{i-1}(J_{i-1}, 0)$, and hence $V_i(0, 1) = V_{i-1}(J_{i-1}, 1)$. We define now $\tilde{V}(j, n)$ for $0 \le j \le J$ as $\tilde{V}(\tilde{j}, n) = V_i(j, n)$ if $\tilde{j}\Delta x = a_i + j\Delta x$, with $1 \le j \le J_i$. With the assumption on f, since $D_x^0 = (D_x^+ + D_x^-)/2$, we can rewrite (45) as

$$(D_x^+ - D_x^- - 2\frac{\Delta x}{\Delta t} D_t^+) \tilde{V}(\tilde{j}_i, 0) + 2\frac{\Delta x}{\Delta t} q(a_i) + \Delta x f(p(a_i), q(a_i), D_x^0 p(a_i)) = 0,$$

with $\tilde{j}_i = J_1 + \ldots + J_i$, which, multiplying by $-\frac{\Delta t}{2\Delta x}$, proves that $\tilde{V}$ is solution of (36) at any point, and therefore U and $\tilde{V}$ coincide at time 0 and 1. A simple recursion with the explicit schemes now proves that, for any n, for any i, $V_i(0, n) = V_{i-1}(J_{i-1}, n)$, and therefore $U = \tilde{V}$. ■

Remark 5.2 *The assumption on f in Theorem 5.1 is fulfilled when $f(u, u_t, u_x) = f_1(u) + f_2(u)u_t + f_3(u)u_x$.*

6 Convergence of the Discrete Algorithm

According to Theorem 5.1, we suppose that f is affine in $\partial_x u$. We introduce the linear transmission operators defined by

$$T^{\pm} = B^{\pm}(0,0), \quad \widetilde{T}^{\pm} = \widetilde{B}^{\pm}(0,0)$$

We note $\bar{U}_i = \bar{U}/_{\Omega_i}$. With these notations, the error $\bar{U}_i^k = U_i^k - \bar{U}_i$ is solution of the linear problem

$$\left(D_t^+ D_t^- - D_x^+ D_x^- + 1\right) \bar{U}_i^k = F_i^k, \quad \text{in } [\![1, J_i - 1]\!] \times [\![1, N]\!] \tag{47}$$

$$\tag{48}$$

with the initial value $\bar{U}_i^k(j,1) = \bar{U}_i^k(j,0) = 0$ and the transmission conditions

$$\begin{aligned} T^- \bar{U}_i^k(0,\cdot) + G_i^{-,k} &= \widetilde{T}^- \bar{U}_{i-1}^{k-1}(J_{i-1},\cdot) + \widetilde{G}_{i-1}^{-,k-1}, \quad \text{in } [\![0,N]\!], \\ T^+ \bar{U}_i^k(J_i,\cdot) + G_i^{+,k} &= \widetilde{T}^+ \bar{U}_{i+1}^{k-1}(0,\cdot) + \widetilde{G}_{i+1}^{+,k-1}, \quad \text{in } [\![0,N]\!]. \end{aligned} \tag{49}$$

The remainders are given by

$$\begin{aligned} F_i^k &= f(U_i^k, D_t^{-*} U_i^k, D_x^0 U_i^k) - f(\bar{U}_i, D_t^{-*} U_i^k, D_x^0 \bar{U}_i) + \bar{U}_i^k \text{ for } n \geq 1, \\ G_i^{-,k} &= -\frac{\Delta x}{2} f_i^k(0,\cdot) + g^-(U_i^k(0,\cdot)) - g(\bar{U}_i(0,\cdot)), \\ \widetilde{G}_{i-1}^{-,k} &= \frac{\Delta x}{2} f_{i-1}^k(J_{i-1},\cdot) + g^-(\bar{U}_{i-1}^k(J_{i-1},\cdot)) - g^-(\bar{U}_{i-1}(J_{i-1},\cdot)), \\ G_i^{+,k} &= -\frac{\Delta x}{2} f_i^k(J_i,\cdot) + g^+(\bar{U}_i^k(J_i,\cdot)) - g^+(\bar{U}_i(J_i,\cdot)), \\ \widetilde{G}_{i+1}^{+,k} &= \frac{\Delta x}{2} f_{i+1}^k(0,\cdot) + g^+(\bar{U}_{i+1}^k(0,\cdot)) - g^+(\bar{U}_{i+1}(0,\cdot)), \end{aligned} \tag{50}$$

and $G_i^{\pm,k}(0) = 0$, $\widetilde{G}_i^{\pm,k}(0) = 0$. For $n = 0$, the centered derivative in x are replaced in the expression of F_i^k by a forward or backward derivative.

We define now a discrete energy as follows. We consider sequences of the form $V = \{V(j)\}_{0 \leq j \leq J}$ in $\mathbb{R}^{J+1}$, and we define a bilinear form on $\mathbb{R}^{J+1}$ by

$$a_\Delta(V,W) = \Delta x\Big(\sum_{j=1}^{J} D_x^-(V)(j) \cdot D_x^-(W)(j) + \sum_{j=1}^{J-1} V(j)W(j)\Big). \tag{51}$$

For a mesh function V of time and space, we define

$$\begin{aligned} E_K(V)(n) &= \frac{\Delta x}{2} \sum_{j=1}^{J-1} \left((D_t^- V(j,n))^2 + (D_t^- V(j,n+1))^2\right), \\ E_P(V)(n) &= a_\Delta(V(\cdot,n), V(\cdot,n-1)), \\ E &= E_K + E_P. \end{aligned} \tag{52}$$

The quantity E_K is a discrete kinetic energy. It is less evident to identify E_P as discrete potential energy. The following lemma gives a lower bound for E under a CFL condition, and hence shows that E is then indeed an energy. The proof is classical ([7]) and is omitted here.

Lemma 6.1 *For any $n \geq 1$, we have*

$$E(V)(n) \geq \left(1 - \frac{\Delta t^2}{\Delta x^2} - \frac{\Delta t^2}{4}\right) E_K(V)(n). \tag{53}$$

Hence, under the CFL condition

$$\frac{\Delta t^2}{\Delta x^2} + \frac{\Delta t^2}{4} < 1, \tag{54}$$

E is bounded from below by an energy.

The following energy estimate is obtained by a discrete integration by parts:

Lemma 6.2 *For any V solution of*

$$\left(D_t^+ D_t^- - D_x^+ D_x^- + 1\right) V = F, \quad 1 \le j \le J-1,\, 1 \le n \le N, \tag{55}$$

we have for any $n \ge 1$,

$$\begin{aligned} E(V)(n) - E(V)(n-1) &+ \frac{\Delta t}{2}[(\widetilde{T}^+ V(0,n))^2 + (\widetilde{T}^- V(J,n))^2] \\ &= \frac{\Delta t}{2}[(T^- V(0,n))^2 + (T^+ V(J,n))^2] + 2\Delta t \Delta x \sum_{j=1}^{J} F(j,n) D_t^0 V(j,n), \end{aligned} \tag{56}$$

and for $n = 0$,

$$\begin{aligned} E_K(U)(0) + E(U)(0) &+ \frac{\Delta t}{4}[(\widetilde{T}^+ U(0,0))^2 + (\widetilde{T}^- U(J+1,0))^2] \\ &= \frac{\Delta t}{4}[(T^- U(0,0))^2 + (T^+ U(J,0))^2] + a_\Delta(P,P) + 2\Delta x \sum_{j=1}^{J} Q(j) D_t^+ U(j,0). \end{aligned} \tag{57}$$

These estimates are obtained by multiplying (55) with $D_t^0 V(j,n)$ and integrating by parts [7]. We now state the main result of this section.

Theorem 6.3 *Suppose that f is affine with respect to $\partial_x u$. Defining the quantities*

$$\begin{aligned} R_i^k(n) = \quad & \widetilde{G}_i^{+,k}(n)(\widetilde{G}_i^{+,k}(n) + 2\widetilde{T}^+ \bar{U}_i^k(0,n)) + \widetilde{G}_i^{-,k}(n)(\widetilde{G}_1^{-,k}(n) + 2\widetilde{T}^- \bar{U}_i^k(J_i,n)) \\ & - G_i^{-,k}(n)(G_i^{-,k}(n) + 2T^- \bar{U}_i^k(0,n)) - G_i^{+,k}(n)(G_i^{+,k}(n) + 2T^+ \bar{U}_i^k(J_i,n)), \end{aligned} \tag{58}$$

and assuming that there exists a positive constant M such that for any iteration number K, any domain Ω_i and any discrete time n, the following estimate holds

$$2\Delta x \sum_{j=1}^{J} F_i^k(j,n) D_t^0 \bar{U}_i^k(j,n) + \frac{1}{2} R_i^k(n) \le M E(\bar{U}_i^k)(n), \tag{59}$$

then, for Δt sufficiently small, the discrete Schwarz algorithm converges in the energy norm.

Proof We apply (56,57) to $\bar{U}_i^k$. Since the initial data vanish, every term in (57) vanish. Thus $E(\bar{U}_i^k)(0) = 0$, and we rewrite (56) as

$$\begin{aligned} E(\bar{U}_i^k)(n) - E(\bar{U}_i^k)(n-1) + \frac{\Delta t}{2}[(\widetilde{T}^+ \bar{U}_i^k(0,n) + \widetilde{G}_i^{+,k}(n))^2 + (\widetilde{T}^- \bar{U}_i^k(J_i,n) + \widetilde{G}_i^{-,k}(n))^2] \\ = \frac{\Delta t}{2}[(T^- \bar{U}_i^k(0,n) + G_i^{-,k}(n))^2 + (T^+ \bar{U}_i^k(J_i,n) + G_i^{+,k}(n))^2] \\ + 2\Delta t \Delta x \sum_{j=1}^{J_i} f_i^k(j,n) D_t^0 \bar{U}_i^k(j,n) + \frac{\Delta t}{2} R_i^k(n). \end{aligned} \tag{60}$$

We now insert the transmission conditions (49), translate the indices in the righthand side, and add the contributions of all subdomains. We define a total internal energy and a total boundary energy as

$$E_I^k(n) = \sum_{i=1}^{I} E(\bar{U}_i^k)(n),\ E_B^k(n) = \frac{\Delta t}{2} \sum_i [(\widetilde{T}^+ \bar{U}_i^k(0,n) + \widetilde{G}_i^{+,k}(n))^2 + (\widetilde{T}^- \bar{U}_i^k(J_i,n) + \widetilde{G}_i^{-,k}(n))^2].$$

With these notations we can write

$$E_I^k(n) - E_I^k(n-1) + E_B^k(n) \le E_B^{k-1}(n) + \sum_{i=1}^{I} [2\Delta t \Delta x \sum_{j=1}^{J_i} f_i^k(j,n) D_t^0 \bar{U}_i^k(j,n) + \frac{\Delta t}{2} R_i^k(n)]. \tag{61}$$

We now sum up (61) for $1 \leq k \leq K$, and define $\hat{E}_I^K(n) = \sum_{k=1}^K E_I^k(n)$:

$$\hat{E}_I^K(n) - \hat{E}_I^K(n-1) + E_B^K(n) \leq E_B^0(n) + \Delta t \sum_{k=1}^{K} \sum_{i=1}^{I} [2\Delta x \sum_{j=1}^{J_i} f_i^k(j,n) D_t^0 \bar{U}_i^k(j,n) + \frac{1}{2} R_i^k(n)]. \tag{62}$$

Under assumption (59) we deduce that

$$\hat{E}_I^K(n) - \hat{E}_I^K(n-1) \leq F_0(n) + M\Delta t \hat{E}_I^K(n). \tag{63}$$

The recursive inequality is easy to solve. For Δt sufficiently small, $M\Delta t < 1$, and since $\hat{E}_K(0) = 0$, we get

$$\hat{E}_I^K(n) \leq e^{Mn\Delta t} \sum_{p=1}^{n} F_0(p) \leq e^{MT} \sum_{p=1}^{N} F_0(p). \tag{64}$$

This proves that $\sup_{n \leq N} \hat{E}_I^K(n)$ is bounded as K tends to infinity. Therefore we have

$$\forall n \leq N, \sum_{i=1}^{I} E(U_i^k - \bar{U})(n) \to 0 \text{ as } k \to +\infty \tag{65}$$

which concludes the proof. ■

We are able to prove the assumption (59) in the case of the linear transmission conditions.

Corollary 6.4 *Suppose that f is affine with respect to $\partial_x u$. For Δt sufficiently small, there exists a time T such that the discrete Schwarz waveform relaxation algorithm (43) with linear transmission conditions (i.e. $g^{\pm} = 0$) converges to the discrete approximation $\bar{U}$ of problem (4) on $(0, T)$.*

Proof Here the remainder $R_i^k(n)$ reduces to

$$R_i^k(n) = \quad 2\Delta x(-f_i^k(0,n) D_t^0 \bar{U}_i^k(0,n) + f_i^k(J_i,n) D_t^0 \bar{U}_i^k(J_i,n)), \tag{66}$$

and the estimate (59) amounts to proving that for any $k \leq K$, $n \leq N$,

$$\Delta x \sum_{j=0}^{J_i} f_i^k(j,n) D_t^0 \bar{U}_i^k(j,n) \leq M_1 E(\bar{U}_i^k)(n). \tag{67}$$

If f is globally Lipschitz in both variables, this is merely an application of the discrete Cauchy-Schwarz lemma. ■

7 Numerical Results

7.1 Remarks on Overlapping versus Nonoverlapping Schwarz Waveform Relaxation Algorithms and Variants

The original Schwarz algorithm uses overlapping domains (domains $\Omega_i = (a_i, b_i)$ for $1 \leq i \leq I$, with $a_i < a_{i+1} < b_i < b_{i+1}$), with an exchange of Dirichlet data on the boundary. It is known for elliptic problems to converge, but the smaller the overlap, the slower the convergence [8]. Due to the finite speed of propagation, it converges in a finite number of iterations, given by $N = \lceil cT/L \rceil$ where c is the wave speed, and L the size of the overlap (see [7] for the linear wave equation). Therefore the convergence can be very slow. In the linear case, using absorbing boundary conditions instead of the Dirichlet transmission, even without overlap, improves drastically the convergence, giving in one dimension a number of iterations equal to 2 for some T [7]. In the nonlinear case, we have theoretical convergence results on the linear algorithm for sufficiently small T. However, we will see that the latter performs very well on a large time interval, and that the nonlinear algorithm performs even better.

Our experiments concern the space domain $\Omega = [0,4]$, simulating $\mathbb{R}$ with linear absorbing boundary conditions at each boundary. The time interval is $[0,2]$. Ω is divided in two subdomains. The initial value is supported in $(0,2)$, with $p(x) = x^3(2-x)^3$, the initial velocity is $q(x) = 3x^2(2-x)^2(x-1)$. This is a good test since the solution is supported in the first subdomain at $t = 0$ and escapes in the second domain before the end of the computation.

Note that the Schwarz algorithm can be viewed as a fixed point algorithm applied to the interface problems

$$(H_1^+, H_2^-) \mapsto (\widetilde{B}^+ U_2(0,\cdot), \widetilde{B}^- U_1(J_1,\cdot)) = \mathcal{A}(H_1^+, H_2^-).$$

In all cases, the stopping criterion in the algorithm will be on the residual res^k for $\mathcal{A}$. We also compute the exact discrete solution in Ω, and measure the discrete global error E^k in $L^\infty(0,T,L^2(\Omega))$.

7.2 The Classical Overlapping Schwarz Algorithm

In this case, $\Omega_1 = (0, 2+L)$ and $\Omega_2 = (2,4)$. The stopping criterion pertains to the residual for the interface problem:

$$res^k = \left(\|(u_1^{k+1} - u_1^k)(2+L,\cdot)\|_{L^2}^2 + \|(u_2^{k+1} - u_2^k)(2,\cdot)\|_{L^2}^2 \right)^{1/2}.$$

We run the computation until the residual res^k is equal to zero. The theoretical minimal number of iterations for the discrete algorithm is $N_{th} = \lceil \frac{\Delta x}{\Delta t} \frac{T}{L} \rceil$.

We start with the nonlinear term $f(u, u_t, u_x) = u^3$. Table 1 gives the number of iterations N_{comp} needed to achieve convergence ($i.e$ the error is zero), together with N_{th}. On the left we choose the overlap equal to 8 grid points, and vary $\Delta t = 1/120$ and $\Delta x = 1/100$ to fulfill the CFL condition. On the right we fix $\Delta t = 1/120$ and $\Delta x = 1/100$, and vary the overlap L.

Δx	1/100	1/200	1/400
Δt	1/120	1/240	1/480
N_{comp}	29	54	105
N_{th}	30	60	120

$L = 8\Delta x$

overlap L	2 Δx	$4\Delta x$	$8\Delta x$	$16\Delta x$
N_{comp}	108	55	28	15
N_{th}	121	61	31	16

$\Delta x = 1/100$, $\Delta t = 1/120$

Table 1: Number of iterations to achieve convergence for the classical Schwarz algorithm

We show on Figure 1 the convergence history of the classical Schwarz algorithm for various values of the mesh size and an overlap equal to eight gridpoints.We check in each case that the error E^k vanishes together with the residual. Furthermore we can see that the error decays very slowly for many iterations, and reaches zero in a few iterations, independently of the mesh size (three or four in all cases). The behaviour is very similar to what happens for the linear wave equation : only the finite speed of propagation produces convergence, which takes place when the signal has left the domain. The algorithm behaves similarly for other nonlinearities.

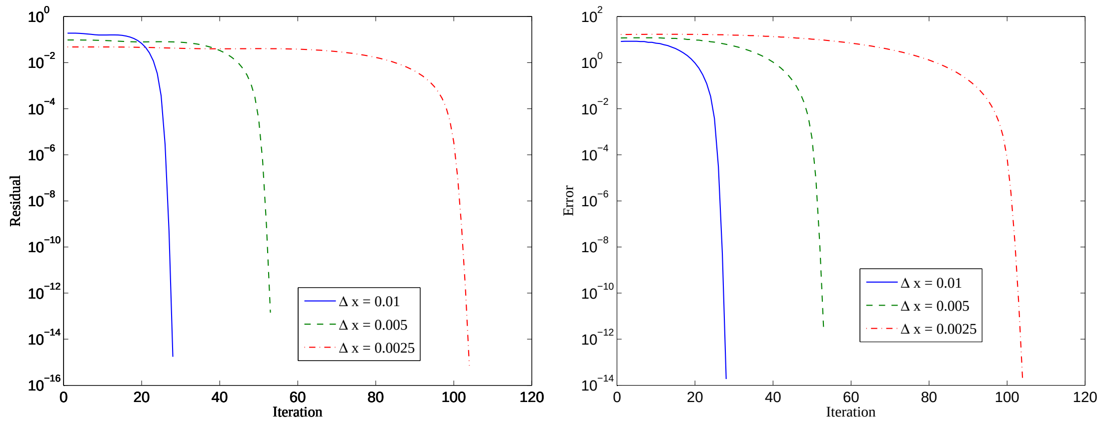


Figure 1: Variation of the residual res^k (left) and the error E^k(right) as a function of the iteration number k.

7.3 The Non linear Nonoverlapping Schwarz Algorithms

We will see in this section that our strategy greatly improves the performances of the Classical Schwarz algorithm. Note that, whereas the classical algorithm only converges in the presence of an overlap, and the smaller the overlap, the slowlier the convergence, our algorithms are run without overlap,

Here the residual is given by:

$$res^k = \left(\|(G^{+,\,k+1} - G^{+,\,k})(a_1,\cdot)\|^2_{L^2} + \|(G^{-,\,k+1} - G_2^{-,\,k})(a_1,\cdot)\|^2_{L^2} \right)^{1/2}.$$

We start with $f = u^3$. In this case the transmission operators are the same and linear, since $g^{\pm} = 0$. We have proved in Corollary 6.4 that there exists a final time T for which the discrete algorithm is convergent.

In the forthcoming computations, the theoretical and numerical data are the same as before. Figure 2 plots the convergence history for various mesh sizes. The computation is stopped as soon as the residual reaches $0.5\,10^{-7}$.

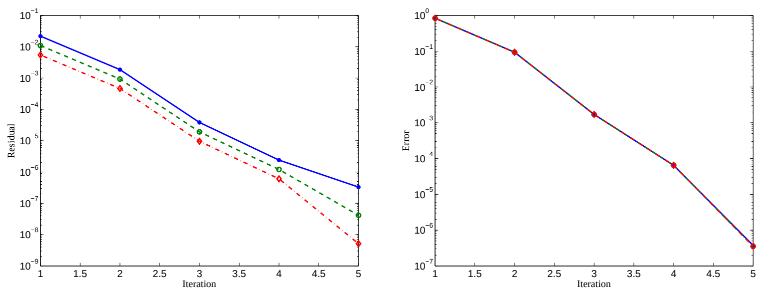
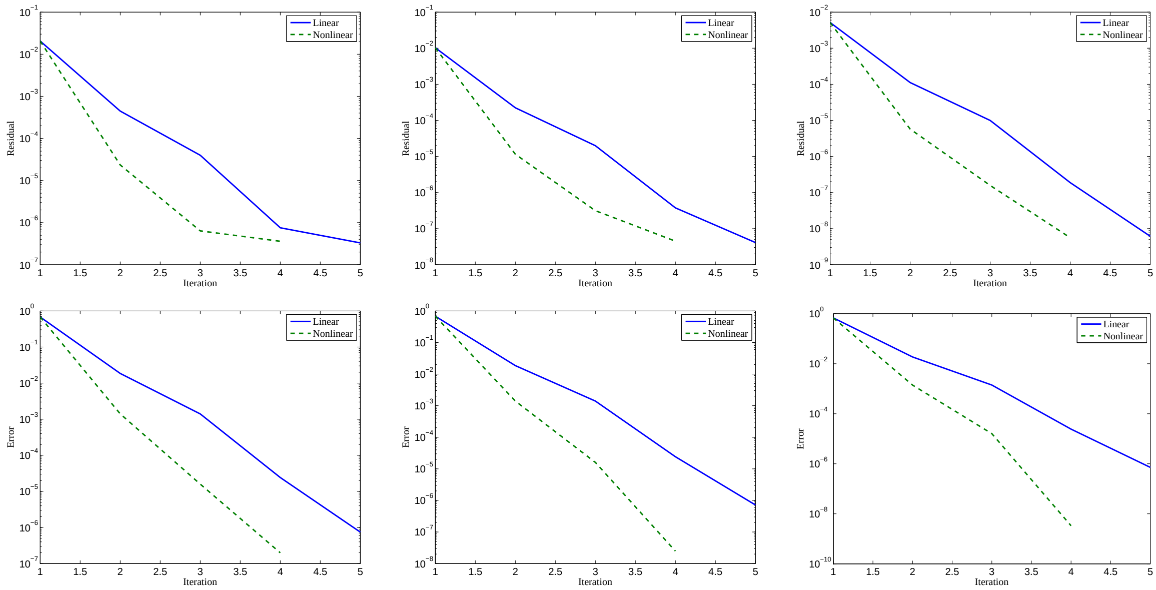


Figure 2: $f = u^3$. Left residual, right: error. $*$: $\Delta x = 1/100$, $\Delta t = 1/120$, o: $\Delta x = 1/200$, $\Delta t = 1/240$, $\diamondsuit$: $\Delta x = 1/200$, $\Delta t = 1/240$.

We see that the algorithm converges very rapidly, independently of the mesh size.

We consider now the case $f = u^2 u_x$, and nonlinear transmission conditions, *i.e.* $g^{\pm}(u) = \pm u^3/6$. In Figure 3, we plot the convergence history for various mesh sizes. The computation is stopped as soon as the residual reaches $0.5\,10^{-7}$.



$\Delta x = 1/100$, $\Delta t = 1/120$ $\Delta x = 1/200$, $\Delta t = 1/240$ $\Delta x = 1/400$, $\Delta t = 1/480$

Figure 3: $f = u^2 u_x$. Top: residual, bottom: error. Solid: linear, dash: nonlinear.

Both the linear and the nonlinear transmission conditions behaves very well. The convergence takes place in 5 iterations with the linear transmission condition, in 4 iterations with the nonlinear one.

We also can vary the nonlinearities in the transmission conditions, we use a real parameter δ, and $g^{\pm} = \pm\delta u^3$. The nonlinear strategy corresponds to $\delta = 1/6$, whereas the linear one is obtained for $\delta = 0$. We draw in Figure 4 the error curves after 3 iterations for the same initial values as before, the mesh sizes are $\Delta x = 1/100$, $\Delta t = 1/120$. We observe that the nonlinear strategy corresponds precisely to the optimal numerical value of the parameter δ, validating the high frequency approach.

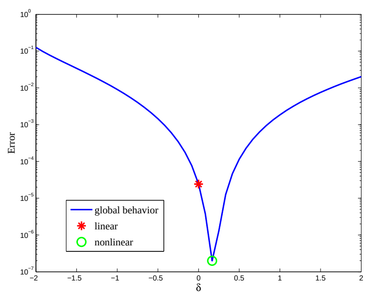


Figure 4: $f = u^2 u_x$. Variations of the error as a function of the nonlinearity coefficient δ

We have carried out the same computations in the case $f = u^2 u_t$. We do not display the results here since they are very similar.

8 Conclusion

We have presented a linear and a nonlinear Schwarz waveform relaxation algorithm without overlap for the semilinear wave equation. On the continuous level, we proved the convergence for sufficiently small time intervals. We designed a discrete algorithm, in such a way that, if convergent, the algorithm converges to the discrete solution in the whole domain, which we prove when the nonlinearity is affine in $\partial_x u$. In that case we proved the convergence for the linear transmission condition. Numerical experiments highlight the fast convergence to the discrete full domain solution in a large time domain in both linear and nonlinear strategies, without overlap. Furthermore, we have shown that our nonlinear transmission conditions give optimal results within a large class of transmission conditions.

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